10

Juggling Patterns, Passing, and Posets

Joe Buhler & Ron Graham
Reed College *UC San Diego*

During the middle of the 1980's a system for describing periodic juggling patterns appeared independently in at least three circles of jugglers. Although given different names by these groups, these patterns have become generally known as "site swaps" and we will use that terminology here.

Site swaps can be described abstractly in a fashion independent of their connection to juggling; they are in fact succinct descriptions of certain kinds of permutations on (one might even say dynamical systems on) the set **Z** of all integers. This gives rise to some pretty mathematics, which we would like to explain in this article.

We'll start with the basic results, which have their roots in permutations of infinite sets and elementary combinatorics. Although these could be described in completely mathematical terms, we both like to juggle, and will use this excuse to persist with the juggling terminology throughout.

Several years ago we had the idea of applying these ideas to patterns with two or more people. Although we never became truly proficient at nontrivial multi-person site swaps (the single person ones are hard enough!) we did discover a broad generalization of the basic counting theorem of site swaps. The result considers permutations of, and colorings of, partially ordered sets (posets), and is contained in the final theorem of this paper. Curiously, we do not yet know a juggling interpretation of this general result.

We warn the reader that mathematics shares one feature with juggling: for maximum enjoyment, you have to try it yourself. In particular, combinatorial arguments tend to be a bit nerve-wracking on first viewing, and real comprehension requires active participation. We encourage the reader in this direction by including exercises, and hope that you tackle them as you read; some of them are answered later in the article, so that if you don't try them right away, you may lose the chance to play along.

Site swaps were discovered independently by Paul Klimek in Santa Cruz, California (in 1982), by Bruce Tiemann and Bengt Magnusson at Caltech (in 1985) and by Colin Wright and other mathematics graduate students in the Cambridge University juggling club (in 1985). Tiemann and Magnusson investigated their theory and (especially) practice and were instrumental in popularizing them within the juggling community. Juggling has many fascinating aspects; however, the discussion here will focus almost exclusively on mathematics, and especially on the ideas necessary to extend site swap theorems to posets. Readers interested in juggling, or its history, or other connections between mathematics and juggling, might want to consult [1], [3], [6], [8], and references included therein. In addition, the web site `www.juggling.org` is a treasure chest of all sorts of information about juggling.

1 Juggling sequences

Imagine the throw times of our juggling balls to be discrete and equally spaced points in time. If a ball that is thrown at time t is next thrown at time t', we draw an arrow from time t to time t'. This is a very idealized view of juggling—it doesn't really matter what the objects are, how they are thrown, or when they are caught, etc. However, as often happens in mathematical models of the real world, sparseness can lead to surprising and acute analyses of underlying patterns.

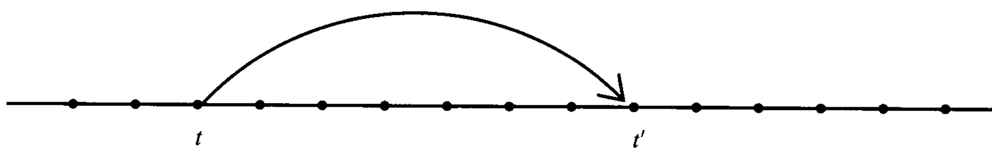

Figure 10.1. One throw

In addition to the arrows themselves, it is useful to annotate each throw with its "height" $t - t'$; roughly, this is how long the ball stays in the air. If no ball is thrown at time t we include a loop from t to t, with height $t - t = 0$. Several examples follow.

In all of these cases, the sequence of throws is periodic, and it is customary to name the trick, as indicated in the diagrams below, by giving the throw heights in a period. The first pattern is the 3-ball cascade (perhaps the most basic juggling pattern), and the second pattern is the 4-ball fountain (waterfall); there is an obvious generalization to the most basic or "canonical" n-ball pattern.

In the standard interpretation of the diagrams, the throws take place alternately from the right and left hands. Although the 3-ball cascade diagram and the 4-ball fountain diagram are similar in appearance, they actually are quite different in the real world. In the cascade a ball is thrown from

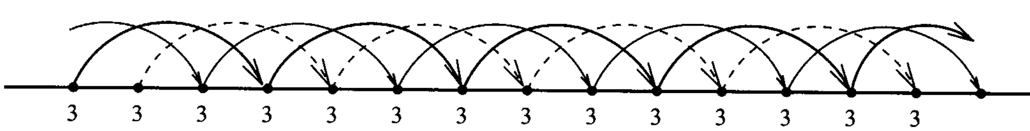

Figure 10.2. $3 = 333\ldots$

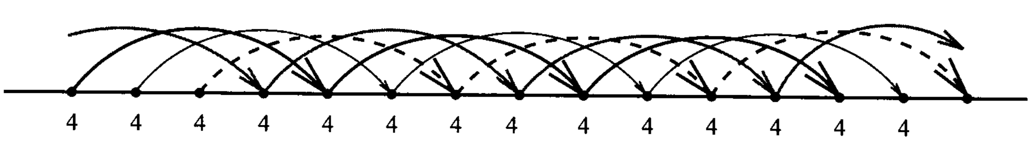

Figure 10.3. 4

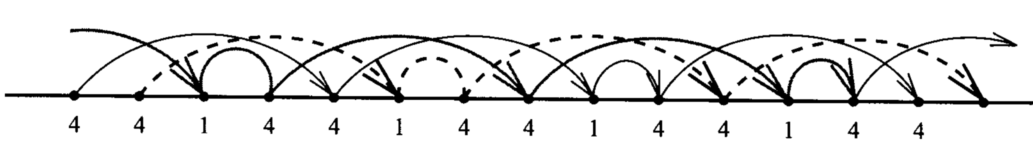

Figure 10.4. 441

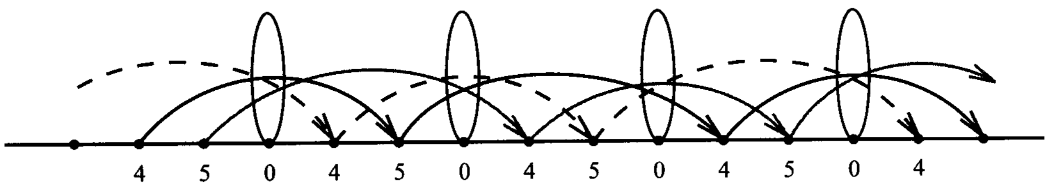

Figure 10.5. 450

one hand to another (since a ball is next thrown 3 time ticks later by the opposite hand), whereas in the fountain balls are thrown from each hand to itself (since a ball is next thrown 4 ticks later by the same hand).

These diagrams can be interpreted mathematically in several ways; we choose to think of them as rightward-moving permutations f of the set

$$\mathbf{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

of all integers. Thus, a juggling pattern determines a function $f: \mathbf{Z} \to \mathbf{Z}$, where $f(t)$ is the next throw-time of the ball thrown at the time $t \in \mathbf{Z}$. From Figure 10.2, we see that the 3-ball cascade "3" is the permutation $f(t) = t + 3$, and the most basic n-ball pattern is $f(t) = t + n$.

We remind the reader that a permutation $f: \mathbf{Z} \to \mathbf{Z}$ is a bijection, which means that it is one-to-one (distinct points are mapped to distinct points), and onto (every u in $\mathbf{Z}$ is in the image). In symbols: if $f(t) = f(t')$ then $t = t'$, and for every u there is a t such that $f(t) = u$.

The permutations in the figures above have the special property that they are rightward-moving, or increasing. This means that $f(t) \geq t$ for all times t.

In the context of juggling, the fact that f is a permutation means that two balls never land at the same time, and that at each instant in time either a ball is caught and thrown (or the hand is empty). If the hand is empty at time t, i.e., no ball arrives or is thrown at time t then $f(t) = t$. It is sometimes mathematically slightly more convenient to disallow these "empty" 0-throws by restricting attention to patterns that satisfy $f(t) > t$. However, the theory of these "positive" or "no empty hands" patterns is essentially the same as the theory that we develop here, since it is easy to check that the relation $g(t) = f(t) + 1$ establishes a one-to-one correspondence between juggling patterns f, and positive juggling patterns g. Since 0-throws do occur in juggling practice we will stick with the convention $f(t) \geq t$.

Exercise. Give an example of a permutation of $\mathbf{Z}$ that is not rightward-moving.
Exercise. How many balls are being juggled in each of the juggling patterns above?
Exercise. Find a juggling pattern that only contains throws with throw heights 7 and 3.

As mentioned above, in the usual two-hand juggling patterns, the odd and even throw times usually correspond to throws with the right and left hands. However, there are many other juggling realizations of a pattern f; for instance, one might imagine a one-handed juggler making all of the throws.

Jugglers really need to know only the throw heights

$$j(t) = j_f(t) = f(t) - t$$

at each time t. We will assume that our patterns are periodic in the sense that j is periodic, i.e., that there is an n such that $j(t+n) = j(t)$ for all t. In particular, the values $j(0), j(1), \ldots, j(n-1)$ uniquely determine j and also determine $f(t) = t + j(t)$.

If t is an integer then let $[t]_n = t \bmod n$ denote the remainder when t is divided by n, i.e., the unique integer a such that $0 \leq a < n$ and $t - a$ is divisible by n. If n is understood, as it

Buhler and Graham check out the theory

often will be, we will denote this by $[t]$. Finally, we let $\overline{n}$ be shorthand for the set of possible remainders $\{0, 1, 2, \ldots, n-1\}$.

Definition 1 A juggling sequence, or site swap, is a finite sequence j of nonnegative integers, usually written as a string

$$j(0)\, j(1)\, \ldots\, j(n-1)$$

such that the mapping $f\colon \mathbf{Z} \to \mathbf{Z}$ defined by

$$f(t) = t + j\big([t]_n\big)$$

is a permutation of the integers.

Formally, a site swap is a function $j: \overline{n} \to \mathbf{Z}$ with nonnegative values, but informally we usually just write a site swap as a string of symbols if the meaning is clear. Thus the site swap "534" is shorthand for the sequence j with $j(0) = 5$, $j(1) = 3$, and $j(2) = 4$.

To actually juggle a juggling sequence j of length n, a juggler just makes throws of height $j([t]_n)$ at time t.

Exercise. Find a finite sequence of nonnegative integers that is not a juggling sequence.
Exercise. Which sequences of length 2 are juggling sequences?
Exercise. Show that $[[a]_n + [b]_n]_n = [a+b]_n$, and $[[a]_n[b]_n]_n = [ab]_n$.

Remark. Tiemann and Magnusson introduced the term "site swap" since they visualized the balls interchanging "sites" in the cyclic ordering of balls of the canonical pattern.

2 When is a sequence a site swap?

For a sequence of nonnegative integers to be a site swap it is clearly necessary that the arrival times $t + j(t)$ be distinct for $0 \le t < n$. Somewhat surprisingly, this isn't sufficient. For instance, the sequence 346 has distinct arrival times for $0 \le t < 3$:

$$0 + j(0) = 3, \quad 1 + j(1) = 5, \quad 2 + j(2) = 8.$$

However, if we look at the corresponding permutation f we see that

$$f(2) = 2 + j(2) = 8 = 4 + j([4]_3) = 4 + j(1) = f(4).$$

Thus there is a "collision" and f isn't a permutation, and 346 isn't a juggling sequence.

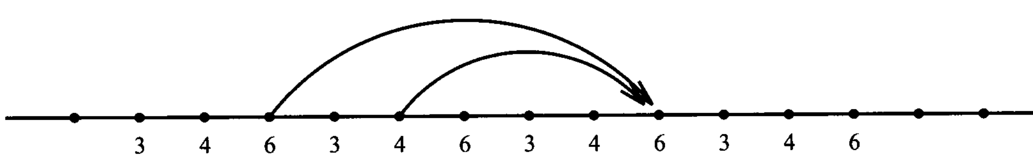

Figure 10.6. A collision

Fix a positive integer n. We show that if a sequence $j(t)$ gives a function $f(t) = t + j([t]_n)$ that has a collision, then already there is a collision modulo n in the first n values. Indeed, suppose that there is a collision, so that $x + j([x]) = x' + j([x'])$ for some x and x'. Writing $x = t + ny$, $x' = t' + ny'$, for $0 \le t,\ t' < n$, we see that $t + j(t) = t' + j(t') + n(y' - y)$. This implies that if f has a collision, then f modulo n has a collision in its first n values. Therefore, if the $t + j(t)$ are distinct modulo n for $0 \le t < n$ then j is a juggling sequence.

Exercise. Show that the converse is true, i.e., that if there are integers t and t' such that

$$t + j(t) \equiv t' + j(t') \bmod n, \qquad 0 \le t, t' < n,$$

then $f(t) := t + j(t)$ isn't a bijection, thereby proving the following theorem.

Theorem 2 *A finite sequence j of nonnegative integers $j(0) \ldots j(n-1)$ is a juggling sequence if and only if the integers $[t + j(t)]_n$, $0 \le t < n$, are distinct.*

Exercise. Show that any cyclic permutation of a juggling sequence is also a juggling sequence.

Exercise. Find a site swap whose reversal, obtained by reading the numbers in reverse order, is not a site swap.

By the theorem, if j is a juggling sequence then the function from $\overline{n}$ to itself defined by

$$\pi_j(t) = [t + j(t)] = f(t) \bmod n$$

is a permutation.

To check whether a sequence of integers is in fact a site swap it is convenient to write t and $j(t)$ in rows, add modulo n, and check whether there are duplications. For instance, one finds that 345 is a juggling sequence, but that 543 is not:

$$\begin{array}{r} 3\ \ 4\ \ 5 \\ +\ \ 0\ \ 1\ \ 2 \\ \hline 0\ \ 2\ \ 1 \end{array} \qquad\qquad \begin{array}{r} 5\ \ 4\ \ 3 \\ +\ \ 0\ \ 1\ \ 2 \\ \hline 2\ \ 2\ \ 2 \end{array}$$

3 How many balls are there?

How many balls are there in the juggling pattern f corresponding to a site swap j?

From looking at the original examples it should be reasonably clear that the number is the number of distinct infinite paths in the diagram. For instance, 3, 450, and 441 are 3-ball tricks, whereas 4 and 345 are 4-ball tricks.

Experiments should convince you that the number of balls is the average throw height. We will prove this assertion by reducing an arbitrary juggling pattern to the basic pattern $f(t) = t + b$, where b is the number of balls, by simple transformations that leave the number of balls and the average throw height unchanged.

Theorem 3 *The number of balls b in a site swap $j(0)\ldots j(n-1)$ is the average*

$$b = \frac{j(0) + \cdots + j(n-1)}{n}.$$

Note that this immediately implies that, for instance, the sequence 344 cannot be a juggling sequence since its average isn't an integer!

To prove the theorem, we start by noting that this formula is obvious if the juggling pattern is $f(t) = t + b$, i.e., the constant site swap

$$b = j(0) = j(1) = \cdots = j(n-1).$$

Now suppose that the sequence is not constant. Then we can find two adjacent terms that are "out of order" in the sense that $j(t) > j(t+1)$. For instance, in the sequence 5744 the 7 and 4 are out of order since the first ball thrown lands after the second ball thrown.

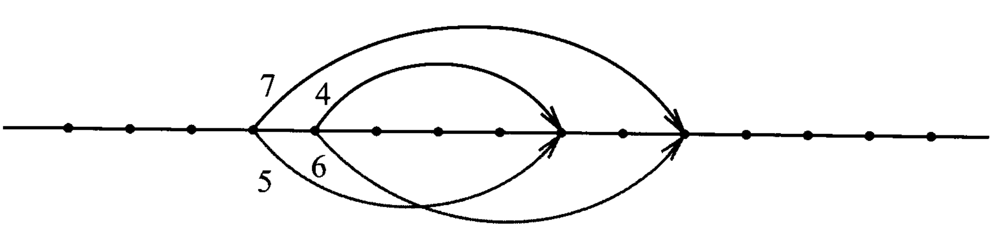

Figure 10.7. Switching two throws

Then we fix the out-of-order pair by interchanging their arrival times. From the above diagram it is clear that 5744 becomes 5564. More generally, if a, b is a consecutive pair with $a > b + 1$ and a is (one of) the largest term(s) in the site swap, then we replace the pair a, b with the pair $b + 1$, $a - 1$. This transformation leaves the number of balls unchanged, leaves the average of the sequence values unchanged, and decreases the largest term or the number of terms equal to the largest term. By continuing to apply this transformation, sooner or later we will arrive at a constant sequence, which has no out-of-order pairs.

In order to carry this out, it may be necessary to remember that a sequence is really a cycle, and count the first and last terms as a consecutive pair. For instance, in the sequence 4457, the first and last terms are out of order, and we apply the above transformation to those two numbers.

In any event, these reductions finally reduce to the basic trick, thereby proving the theorem.

Exercise. How many such steps are needed to convert 41479 to 66666 by this procedure?
Exercise. List all 3-ball juggling sequences of period 3.
Exercise. Find a sequence $j(0)\,j(1)\,j(2)\,j(3)$ of length 4 that is not a juggling sequence, but nonetheless has distinct $t + j(t)$, and whose average is an integer.
Exercise. Show that any sequence of integers whose average is an integer can be rearranged to be a juggling sequence. (Warning: this problem seems to be very hard; the only proof that we know is based on a paper by the famous group theorist Marshall Hall from 1952; his paper was on abelian groups and predates site-swaps by 30 years.)

4 How many site swaps are there with b balls and period n?

This question is much more involved than the earlier ones, and will take several sections to answer.

If j is a juggling sequence of length n then $\pi(t) = [t + j(t)]$ is a permutation of $\overline{n}$.

It is well known that there are $n!$ permutations of an n-element set. (There are n choices for the image of the first element, $n - 1$ for the second elements, and so on, giving a total of

$$n \cdot (n-1) \cdots 2 \cdot 1 = n!$$

permutations.) The collection of all permutations of $\overline{n}$ is usually denoted S_n and is called the symmetric group on n objects.

Exercise. Find the permutation π of $\overline{5}$ associated with the juggling sequence 41479.

For permutations π, let $\mathbf{JS}(\pi, b)$ denote the number of juggling sequences with b balls whose associated permutation is π, and let $\mathbf{JS}(n, b)$ denote the number of all juggling sequences of period n with b balls. Therefore

$$\mathbf{JS}(n,b) = \sum_{\pi} \mathbf{JS}(\pi, b)$$

where the sum is over all π in S_n.

To construct a juggling sequence whose associated permutation is π we can more or less reverse the procedure used to find π from j. That is, we first write $\pi(t)$ and t in rows and subtract to get a row $\pi(t) - t$. The new row sums to 0 (and could loosely be interpreted as a 0-ball juggling pattern in which one is allowed to throw balls backwards in time!).

Exercise. Prove that the numbers $\pi(t) - t$, $0 \le t < n$, sum to 0.

For example, for the permutation π of $\overline{3}$ that interchanges 1 and 2 we get

$$\begin{array}{rrrrl} & 0 & 2 & 1 & \qquad \pi(t) \\ - & 0 & 1 & 2 & \qquad t \\ \hline & 0 & 1 & -1 & \qquad \pi(t) - t \end{array}$$

This is clearly not a site swap since it has a negative term. However, if we add n's (where here $n = 3$) to make all entries nonnegative, then we will have a site swap. If we want a b-ball site swap then we must add b different n's (since the original sum is 0).

To summarize: all possible b-ball juggling sequences with permutation π are obtained by adding n to entries of $\pi(t) - t$ subject to two conditions: every entry must ultimately be nonnegative, and n is added b times.

In the example above for $n = 3$, we have to add 3 to the entries. Clearly we have to add at least one 3 since the last entry is negative. There is precisely one way to do this to get a 1-ball juggling sequence: add 3 to the last entry to get 012.

There are three ways to add two 3's to get 2-ball juggling sequences, since we add one 3 to the last entry and can add the other to any of the three entries; the three 2-ball juggling sequences corresponding to π are 312, 042, and 015.

Exercise. How many 3 and 4-ball sequences are associated with this specific π?
Exercise. How many 3 and 4-ball sequences are associated with the permutation π that has $\pi(0) = 2, \pi(1) = 0, \pi(2) = 1$?

Clearly the problem of calculating $\mathbf{JS}(\pi, b)$ depends on how many negative integers there are in the row $\pi(t) - t$. Specifically, for each t where $\pi(t) - t$ is negative we are required to add at least one n.

Definition 4 Let $\pi \in S_n$ be a permutation of $\overline{n}$. Then an element t, $0 \leq t < n$, is a **drop** of π if $\pi(t) < t$.

Exercise. How many permutations in S_3 have 0 drops? 1 drop? 2 drops? 3 drops?

Suppose that $\pi \in S_n$ has k drops. To make a site swap of period n and b balls that is associated with π we start with the row $\pi(t) - t$, and add an n at each of the k locations of a drop; after this, every entry is nonnegative. At this point, we choose $b - k$ arbitrary locations (repetitions allowed) and add an n at those locations.

Theorem 5 *Suppose that $\pi \in S_n$ has k drops. Then:*

$$\mathbf{JS}(\pi, b) = \binom{b - k + n - 1}{n - 1}.$$

The binomial coefficient in the theorem seems a bit unlikely. Recall that the binomial coefficient

$$\binom{r}{s} = \frac{r!}{s!(r-s)!} = \frac{r(r-1)\cdots(r-s+1)}{s!}, \qquad 0 \leq s \leq r$$

(read "r choose s") is the number of ways of choosing s things from a set of r things, with repeated choice not allowed and the order of choice ignored; this is the same as the number of r-element subsets of an s-element set. (*Proof.* There are r choices for the first element, $r - 1$ for the second, and $r - s + 1$ for the sth element; each subset can be ordered in $s!$ ways, so if S is the number of subsets then $s!S = r(r-1)\cdots(r-s+1)$.)

As you may already know, binomial coefficients can be constructed recursively "row by row," and written in an especially pleasing format called Pascal's Triangle.

Exercise. What fraction of 5-card poker hands are "flushes" (contain cards from only one suit)? What fraction of 5-card poker hands are "straights" (five consecutive cards)? Which hand is ranked more highly in poker, and why?

Exercise. Show that the binomial coefficients are "symmetric" in the sense that

$$\binom{r}{s} = \binom{r}{r-s}$$

in two ways: algebraically, using the formula above, and combinatorially, by arguing that the number of subsets of an r-element set with s elements is the same as the number of subsets with $r-s$ elements.

In any event, the combinatorial interpretation of the binomial coefficient on right-hand side of the theorem doesn't seem to have much to do with juggling sequences. First, note that if the permutation has 0 drops (i.e., is the identity permutation $\pi(t) = t$), then the theorem asserts that the number of ways to choose b things out of n, with repetition allowed (and order irrelevant), is

$$\binom{b+n-1}{n-1}.$$

More generally, by our earlier remarks, we have to count the number of ways that we can choose $b-k$ objects out of n, with repetition allowed (and order does not matter).

Somewhat surprisingly, the binomial coefficient can be interpreted in this fashion; this combinatorial interpretation is standard, but not nearly as well-known as the usual interpretation. In order to model the case in which repetition is allowed but the order does not matter, we consider nondecreasing sequences of not-necessarily-distinct integers, such as

$$112444$$

in which six objects—two 1's, one 2, and three 4's—have been chosen from $\{1,2,3,4\}$.

Lemma 1 *The number of nondecreasing sequences of length r chosen from $\{1,\ldots,s\}$ is*

$$\binom{s+r-1}{r-1} = \binom{s+r-1}{s}.$$

Proof. The equality between the two formulas is just the symmetry of the binomial coefficients in a previous exercise.

To show that the number of sequences is given by the binomial coefficient, encode each sequence by replacing numbers by stars, separating each transition from i to $i+1$ with a bar. For instance, the selection of $1,1,2,4,4,4$ from $\{1,2,3,4\}$ is encoded

$$* * \mid * \mid\mid * * *$$

A little thought reveals that there is a one-to-one correspondence between the stated sequences and these encodings. The encodings contain $r+s-1$ terms (s stars, and $r-1$ bars separating them), and the number of such encodings is therefore the binomial coefficient stated in the theorem: the number of ways to choose the bars (or stars) in $s+r-1$ positions. □

Let $\delta_n(k)$ denote the number of elements of S_n that have k drops. The number $\mathbf{JS}(n,b)$ of juggling sequences of length n and b balls is the sum of $\mathbf{JS}(\pi,b)$ over all permutations π, so we immediately obtain the following corollary of the theorem.

Corollary 6

$$\mathbf{JS}(n,b) = \sum_{k=0}^{n-1} \delta_n(k)\binom{b-k+n-1}{n-1}.$$

For instance, for $n = 3$ one finds by examination of all six permutations that $\delta_3(0) = 1$, $\delta_3(1) = 4$, $\delta_3(2) = 1$. Thus

$$\mathbf{JS}(3, b) = 1 \cdot \binom{b+2}{2} + 4 \cdot \binom{b+1}{2} + 1 \cdot \binom{b}{2} = 3b^2 + 3b + 1.$$

Exercise. Verify that $\mathbf{JS}(4, b) = (b+1)^4 - b^4$.

5 Drops versus Descents

To go further we need to be able to say something about $\delta_n(k)$. To do this, we have to consider a superficially unrelated concept.

Definition 7 An integer t, $0 \le t < n-1$, is a **descent** of a permutation

$$\pi : \quad \pi(0)\,\pi(1)\,\ldots\,\pi(n-1)$$

if $\pi(t) > \pi(t+1)$. The number of permutations in S_n with k descents is denoted

$$\left\langle {n \atop k} \right\rangle .$$

These numbers are called Eulerian numbers [4] and arise in many different contexts.

Here is a table of the drops and descents of 3-object permutations.

π			# drops	# descents
1	2	3	0	0
1	3	2	1	1
2	1	3	1	1
2	3	1	1	1
3	1	2	2	1
3	2	1	1	2

Exercise. Make an analogous table for $n = 4$.

Although the number of drops of a permutation is not the same as the number of descents, the total number of permutations with k drops is the same as the total number of permutations with k descents.

Theorem 8 *For all positive integers n and k, with $0 \le k < n$,*

$$\delta_n(k) = \left\langle {n \atop k} \right\rangle .$$

The proof of this assertion involves a curious bijection between permutations π with k drops and permutations σ with k descents.

First, we note that a permutation π can be written as a product of "cycles." For example a permutation $\pi \in S_6$ could be given in "two-row" notation as

i	0	1	2	3	4	5
$\pi(t)$	4	5	0	3	2	1

The theory works!

It could also be succinctly described in row-form as 450321. Note that π takes 0 to 4, 4 to 2, and 2 back to 0, and takes 1 to 5 and 5 back to 1. The corresponding cycle notation is $(024)(15)(3)$. The one-cycle (3) corresponds to a fixed point, and is often omitted, but for our purposes below we need to explicitly include 1-cycles.

We define a map $\pi \mapsto \sigma$ from S_n to itself as follows: Write π as a product of cycles, with each cycle written with its largest element at the beginning, and the cycles written in the order in which their largest elements are increasing. Then erase the parentheses to get the row-form of a permutation σ.

For instance from the permutation $\pi = 450321$ above we get the "canonical" cycle form $(3)(402)(51)$ and hence σ has row-form 340251.

Exercise. Let π be the permutation that reverses the row $0 \ldots n-1$, i.e., $\pi(t) = n-1-i$. Describe the corresponding σ.

Note that π can be reconstructed uniquely from σ. Indeed, from the row-form of σ we start with a left parenthesis, and then close the cycle, and begin another, exactly when we see a new record high-point in the string of integers. Thus 340251 gets mapped to $(3)(402)(51)$ as desired.

A little thought confirms that this peculiar procedure exactly undoes the original mapping; since we have mutually inverse mappings from S_n to itself we see that this correspondence is bijective.

Further thought shows that the drops of π correspond to the descents of σ; in particular if π has k drops then σ has k descents. Indeed, any drop $\pi(t) < t$ shows up as consecutive "out-of-order" numbers in the cycle form of π and hence is a descent in σ. On the other hand, any descent in σ must occur within a cycle of π (since the transitions from one cycle to another are increases), and hence corresponds to a drop of π.

Exercise. Prove that

$$\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n-k-1} \right\rangle .$$

(Hint: consider the relationship between the descents of a permutation and the permutation obtained by reversing its row-form.)

Exercise. Give a combinatorial proof of the basic rule of formation of Pascal's Triangle:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

(Hint: Let a be a specific element of an $n+1$-element set, and consider k-subsets that, respectively, do and do not contain a.)

Exercise. Give a combinatorial proof of the following identity:

$$\left\langle {n \atop k} \right\rangle = (k+1) \left\langle {n-1 \atop k} \right\rangle + (n-k) \left\langle {n-1 \atop k-1} \right\rangle .$$

6 Worpitzky's identity

An important formula, sometimes called Worpitzky's identity ([4]), gives a relationship between the Eulerian numbers and perfect powers, and has a close relationship to the formula given earlier for the number $\mathbf{JS}(n, b)$ of site swaps with b balls.

Theorem 9 *If n and b are nonnegative integers then*

$$b^n = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \binom{b+k}{n}.$$

All of the terms in the theorem have combinatorial interpretations, and it is natural to ask for a combinatorial proof; such a proof is given below. We have been unable to locate a combinatorial proof in the literature, but it seems likely that it can be found somewhere.

Before giving the proof, we observe that the identity implies a slick formula for the number of site swaps with fewer than b balls. Indeed, if $\mathbf{JS}_<(n, b)$ denotes the number of patterns with period n with fewer than b balls, then Corollary 6, and the relationship between $\delta_n(k)$ and Eulerian numbers, gives

$$\begin{aligned}
\mathbf{JS}_<(n,b) &= \sum_{a=0}^{b-1} \mathbf{JS}(n,a) \\
&= \sum_{a=0}^{b-1}\sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \binom{a-k+n-1}{n-1} \\
&= \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \sum_{a=0}^{b-1} \binom{a-k+n-1}{n-1}.
\end{aligned}$$

By induction using the recursion for binomial coefficients, or purely combinatorially, or otherwise, one can show that

$$\sum_{i=0}^{m} \binom{i+n-1}{n-1} = \binom{m+n}{n}.$$

Exercise. Write the first 6 rows of Pascal's triangle, and circle the entries involved on the left-hand side of the identity for $m = n = 3$. Explain how the identity follows from the law of formation of Pascal's triangle. Prove the identity in general.

After a little algebraic juggling (replacing $a - k$ by i and letting i run from 0 to $b - k - 1$) we find that we now have

$$\mathbf{JS}_<(n, b) = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \binom{n+b-k-1}{n}.$$

Changing variables by replacing k by $n - k - 1$ gives

$$\mathbf{JS}_<(n, b) = \sum_{k=0}^{n-1} \left\langle {n \atop n-k-1} \right\rangle \binom{b+k}{n}.$$

Using the symmetry

$$\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n-k-1} \right\rangle$$

of the Eulerian numbers and Worpitzky's identity

$$b^n = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \binom{b+k}{n}$$

we arrive at our main counting theorem.

Theorem 10

$$\mathbf{JS}_<(n, b) = b^n.$$

Remarks: The proof of this formula has taken us through a grand tour of parts of combinatorics. However, the elegance of the result suggests that one can hope for a purely combinatorial ("bijective") proof of the result, in which juggling sequences of length n and fewer than b balls are put in one-to-one correspondence with words of length n with symbols chosen from an alphabet with b letters. In fact, several people have constructed such bijections; see the appendix in [3] and also the references cited therein. However, the proof chosen here provides a perfect warm-up for the generalizations to posets. Also, the bijective proof of Worpitzky's identity in the next section could in principle be combined with the earlier arguments to give a bijective proof of the theorem.

From the fact that $\mathbf{JS}(n, b) = \mathbf{JS}_<(n, b+1) - \mathbf{JS}_<(n, b)$ we immediately deduce a formula for $\mathbf{JS}(n, b)$ from the main theorem.

Corollary 11

$$\mathbf{JS}(n, b) = (b+1)^n - b^n.$$

We note that our definitions count cyclic permutations of a site swap as being different from each other. A juggler might not care to make this distinction, and it is easy to use "Möbius inversion" to give a formula for the number of patterns of length n with b balls "up to cyclic permutation" (see [3]).

Exercise. Look up Möbius inversion in your favorite combinatorics book, and find an explicit formula for the number of patterns of length n with b balls "up to cyclic permutation."

7 A combinatorial proof of Worpitzky's identity

Algebraic proofs of Worpitzky's identity

$$b^n = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \binom{b+k}{n}$$

are possible, but in keeping with the spirit of this note, we give a combinatorial proof. The term b^n on the left-hand side of the identity counts the number of n-tuples where each component comes from a b-element alphabet. Thus we must exhibit an explicit one-to-one-correspondence between n-tuples of integers $(x_1, \ldots, x_n)$, $0 \leq x_i < b$, and objects counted by terms in the sum on the right-hand side of the identity.

Given an n-tuple $(x_1, \ldots, x_n)$ of integers between 0 and $b-1$, we determine a unique permutation σ of $\{1, 2, \ldots, n\}$ by the conditions that

$$x_{\sigma(1)} \leq x_{\sigma(2)} \leq \cdots \leq x_{\sigma(n)}$$

and equality $x_{\sigma(i)} = x_{\sigma(i+1)}$ occurs if and only if $\sigma(i) > \sigma(i+1)$.

For example, for $n = 3$ there are $3! = 6$ permutations; in the following table each permutation is followed by a "template" that describes the conditions on the x_i that give rise to that permutation according to the above rule. The key facts are that the "$\leq$" possibilities occur exactly at the descents of the corresponding permutation, and that any σ corresponds to exactly one template.

σ	template
123	$x_1 < x_2 < x_3$
132	$x_1 < x_3 \leq x_2$
213	$x_2 \leq x_1 < x_3$
231	$x_2 < x_3 \leq x_1$
312	$x_3 \leq x_1 < x_2$
321	$x_3 \leq x_2 \leq x_1$

Exercise. Show that there are $\binom{b}{3}$ ways to choose x_i in the first template, i.e., integers x_i between 1 and b such that $x_1 < x_2 < x_3$.

Exercise. Show that there are $\binom{b+2}{3}$ ways to choose x_i in the last template in the table.

Exercise. Show that there are $\binom{b+1}{3}$ ways to satisfy each of the other four templates, and verify that the total number of ways to fill in the templates is b^3.

Exercise. How many ways are there to fill in the template

$$x_1 \leq x_2 < x_3 \leq x_4$$

with integers from 1 to b?

Given a permutation σ, and the associated template, how many ways can one fill in the template with x_i? If σ has k descents then there are k possible equalities in the template; to fill in the template we must choose m of those k locations where we will actually have equality, and then choose $n - m$ distinct elements of $\{1, \ldots, b\}$ to fill into the template. In other words, we choose a total of n objects out of the set

$$\{1, \ldots, b\} \cup D,$$

where D is the k-element set of descent locations. There are $\binom{b+k}{n}$ ways to choose n objects out of $b+k$.

Therefore the total number of all n-tuples $(x_1, \ldots, x_n)$ is

$$\sum_{\sigma} \binom{b+k_\sigma}{n},$$

summed over all permutations σ, where k_σ is the number of descents of σ. Collecting terms with the same number, k, of descents shows that the grand total is given by the sum on the right-hand side of Worpitzky's identity, i.e.,

$$b^n = \sum_{\sigma} \binom{b+k_\sigma}{n} = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \binom{b+k}{n}.$$

This proves the identity for positive integers b. Since both sides of the identity are polynomials in b, and the polynomials are equal for all positive integers b, it follows that the two polynomials are equal.

8 Posets

Suppose that two jugglers are allowed to throw objects at each other. More precisely, suppose that their throw times coincide and that they can throw a ball either to themselves or their partner. Then a juggling pattern (without collisions) could be viewed as a rightward moving permutation of a double-line.

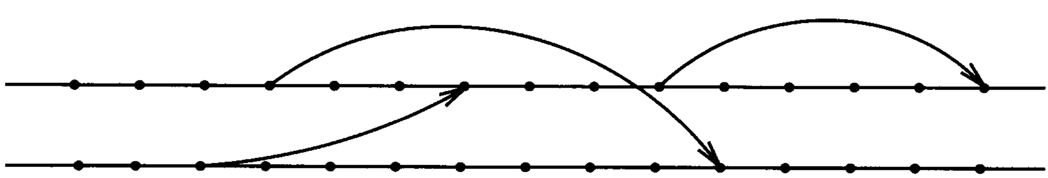

Figure 10.8. A double chain

Note that two throws are made at every time, since each juggler makes a throw. If we require that the pattern be periodic, then we can imitate the previous analysis and look at permutations of a finite "double-chain" of length n and with $2n$ positions.

We carried this analysis through and were surprised to find that the total number of b-ball patterns of period n again had an elegant description: there are $b^n(b-1)^n$ patterns with b objects of length n. The simplicity of this surprised us. What's going on?

After some further experimentation, we discovered that the result extends to a much more general situation, involving partially ordered sets (posets). However, as noted earlier, we don't as yet have a juggling interpretation for permutations on posets.

A **poset** is a set on which there is a relation $<$ that is transitive ($x < y$ and $y < z$ imply that $x < z$), irreflexive ($x < x$ is false), and anti-symmetric (at most one of $x < y$ and $y < x$ is true). The relation $<$ on P has all of the properties of less-than on numbers except that for given x and y in P we do not require that one of $x < y$ and $y < x$ always be true.

One example is the finite "chain" consisting of the integers $1, \ldots, n$ with the usual $<$ relation; in this case every pair of distinct elements is comparable, i.e., for $x \neq y$, either $x < y$ or $y < x$.

Another example is the set of all subsets of the n-element set $\overline{n}$ where the $<$ relation is "is a proper subset of". There are 2^n elements of this poset, and many pairs of elements are not comparable.

If P is a finite poset, then its **incomparability graph** G_P has the elements of P as vertices, and there is an edge joining vertices x and y if and only if x and y are incomparable in P, i.e., neither $x < y$ or $y < x$ is true.

If P is the n-element chain given above, then G_P has n vertices and no edges. If P is the set of subsets of $\overline{n}$, under strict inclusion, then an edge joins two subsets if neither is contained in the other.

Let P be a finite poset. If π is a permutation of the elements of P, then an element x is a **drop** of π if $\pi(x) < x$. We let $\delta_P(k)$ be the number of permutations π of P that have k drops.

Exercise. Let P be the double-chain of length 3. In other words, P has elements $\{0, 1, 2, 0', 1', 2'\}$ with the order relation induced by the usual relation on the underlying integers. Thus i and i' are incomparable, but every other pair of elements x, y satisfies $x < y$ or $y < x$. How many permutations of P have k drops, for $k = 0, 1, 2, 3, 4, 5$?

By analogy with the case of a finite chain of length n, and the two-juggler analysis referred to above, we are led to consider the expression

$$\sum_k \delta_P(k) \binom{b+k}{n}.$$

In order to evaluate this we need a new idea!

9 Graph Colorings

A legal coloring of a graph G is a mapping $\lambda\colon G \to \mathbf{Z}^+$ from the vertices of G to positive integers such that adjacent vertices are assigned different values. (Vertices of a graph are said to be adjacent if they are connected by an edge of the graph.) Usually, the values of λ are thought of as "colors", so the condition $\lambda(v) \neq \lambda(v')$ when there is an edge between v and v' says that adjacent vertices have different colors.

If the possible colors are restricted to $\{1, \ldots b\}$, then λ is said to be a b-coloring. If G is a graph then we let $\chi_G(b)$ denote the number of legal b-colorings of G. It turns out that the function $\chi_G(b)$ is always a polynomial in b, and this first arose in the literature nearly a hundred years ago in connection with attempts to prove the celebrated 4-color map-coloring conjecture (which is now a theorem!).

The goal of this section is to prove that the sum mentioned at the end of the previous section is given by the value, at b, of the chromatic polynomial of the incomparability graph G_P of the poset P, i.e., that

$$\sum_k \delta_P(k) \binom{b+k}{k} = \chi_{G_P}(b)\ .$$

Exercise. Verify that if P is the double-chain of length n referred to above then $\chi_{G_P}(b) = b^n(b-1)^n$.

To prove this we follow the earlier outline: we count legal b-colorings of G_P.

First we need to make one technical remark in order to clarify later notation. We explicitly number the elements of P in such a way that if the ith vertex is less than the jth vertex in P then $i < j$ (but not necessarily conversely). This is sometimes called a **linear extension** of the partial order on P ([7, p. 110]).

Exercise. Prove that every partial order on a finite set can be extended to a complete order. (An order relation "$<$" on a finite set is **complete** if for all x and y either $x < y$ or $y < x$ is true.)

For notational convenience we will think of P as $\{0, \ldots, n-1\}$, but this means that when we refer to an order on P we have to clarify whether this refers to the original partial order on P, or the stronger linear order on the corresponding integers; we will refer to the latter as the "total" order on P.

Let λ be a coloring of the elements of P that gives a legal b-coloring of the incomparability graph G associated with a poset P, i.e., for all x, $1 \le \lambda(x) \le b$, and for all x and y, if x and y are incomparable in P then $\lambda(x) \ne \lambda(y)$.

We will now show that λ determines a unique permutation $\sigma = \sigma(\lambda)$ of $\{0, \ldots, n-1\}$ in a manner that exactly generalizes our earlier construction.

Let σ be a permutation such that

$$(1) \qquad \lambda(\sigma(0)) \le \lambda(\sigma(1)) \le \cdots \le \lambda(\sigma(n-1)),$$

and if $\sigma(i)$ and $\sigma(i+1)$ have the same color, i.e., $\lambda(\sigma(i)) = \lambda(\sigma(i+1))$, then

$$(2) \qquad \sigma(i) > \sigma(i+1) \text{ in } P.$$

A little thought shows that this ordering of the vertices is uniquely determined: inside any run of vertices with the same color our definitions imply that the vertices are linearly (completely) ordered as elements of P, and (2) requires that the order within a run reverses the order in P, so that the vertices are ordered in a uniquely specified fashion, and σ is well-determined.

Thus to every legal coloring λ we have associated a permutation σ of P. We say that i is a **descent** for σ if $\sigma(i) > \sigma(i+1)$ in P.

Note that the notion of a descent depends on the choice of a linear extension of the partial order on P, whereas the notion of a drop depends only on the partial order itself; in this sense the notion of a drop seems more fundamental. However, the basic fact remains the same.

Lemma 2 *The number of permutations σ of P with k descents is equal to the number of permutations π with k drops.*

Proof. The earlier argument works. Briefly, given π we construct σ by writing π in cycle form, putting the largest element at the beginning of each cycle, and ordering by cycles so that their first elements increase. (Here we use the total order on P coming from the integers.)

Given σ we build π by forming cycles, closing a cycle when we see a larger number than any that we've seen before. (Again, using the total order on P.)

One then checks as before that these constructions are mutually inverse.

Moreover, if $j = \sigma(i)$, $k = \sigma(i+1)$, and $j > k$ (using the order on P), then jk must be internal to a cycle of π, so that $j > k = \pi(j)$; the converse is easily checked. Thus drops of π correspond exactly to descents of σ. □

Strictly speaking, this result is not needed for the proof of the theorem, but it does show that the number of permutations with a given number of descents is independent of the choice of linear extension, i.e., that it is intrinsic to the original partial order.

As in the earlier case, a permutation σ determines an "inequality template," and the number of colorings of the template is $\binom{b+k}{n}$ if σ has k descents. All in all, we have proved our main theorem for posets.

Theorem 12

$$\sum_k \delta_P(k) \binom{b+k}{n} = \chi_{G_P}(b).$$

Taking $b = -1$, we see that the value $\chi_G(-1)$ of the chromatic polynomial is equal to $(-1)^n\delta_P(0)$. By a classic theorem of Stanley, $(-1)^n\chi_G(-1)$ is the number of "acyclic orientations" of the graph G, i.e., the number of ways to choose a direction for the edges of G so that there are no directed cycles. Thus drop-free permutations (which certainly sound like something that jugglers should like) correspond to acyclic orientations. It is natural to ask for a bijective proof of this result. Lemma 3 implies that it suffices to find a bijection between descent-free permutations and acyclic orientations.

Exercise. Let L be a descent-free permutation of P, written in the form of a list of all n elements of P in which no element is larger than its successor. Construct an orientation θ of the graph G by orienting an edge $\{x, y\}$ from x to y exactly when x occurs earlier than y in the list L. Show that θ is acyclic. Show that L can be uniquely recovered from θ as follows: let the first element of L be the smallest source in G. (A vertex is a source if it has no incoming arrows; you need to verify that in fact there is a smallest source.) Remove this element from P and G, and apply the same procedure to the smaller poset to determine the second element of the list. Etc.

Exercise. Let P be the union of two chains of length 2, i.e., the poset on $\{a, b, c, d\}$ whose only relations are $a < b$ and $c < d$. Explicitly identify the drop-free and descent-free permutations of P, and the acyclic orientations of G, and identify how they correspond under the preceding bijections.

Acknowledgment: The authors would like to thank Peter Doyle and Oliver Schirokauer for helpful comments on this paper.

References

[1] J. Buhler and R. Graham, Fountains, Showers, and Cascades, *The Sciences*, Jan-Feb, 1984, 44–51.

[2] ——, A note on the binomial drop polynomial of a poset, *Journal of Combinatorial Theory, Series A* **66** (1994), 321–326.

[3] J. Buhler, D. Eisenbud, R. Graham, and C. Wright, Juggling Drops and Descents, in *Organic Mathematics*, 133–154, CMS Proc., **20**, AMS, Providence, RI; appeared originally, without the appendix, in *Amer. Math. Monthly* **101** (1994), 507–519.

[4] R. Graham, D. Knuth, and O. Patashnik, *Concrete Mathematics*, Addison Wesley, 1989.

[5] Marshall Hall, Jr., A combinatorial problem on abelian groups, *Proc. Amer. Math. Soc.* **3** (1952), 584–587.

[6] A. Lewbel and P. Beek, The Science of Juggling, *Scientific American*, November 1995, **273**, 92–97.

[7] R. Stanley, *Enumerative Combinatorics*, Volume I, Cambridge University Press, 1997.

[8] M. Truzzi, On keeping things up in the air, *Natural History*, December 1979, 44–55.